113 Class Problems: Sets and Functions

1. Let S and T be two sets with $f : S \to T$ a function. Under what circumstances does the following statement hold?

$$f \text{ injective} \iff f \text{ surjective} \iff f \text{ bijective} .$$

Carefully justify your answer.

Solution: Assume $|S| = |T| < \infty$

f injective
$\Rightarrow |S| = |\mathrm{Im}(f)|$
$\Rightarrow |\mathrm{Im}(f)| = |T|$
$\Rightarrow \mathrm{Im}(f) = T$

Trivial

$$\begin{array}{ccc} & f \text{ injective} & \\ \Uparrow & & \Downarrow \\ f \text{ bijective} & \Leftarrow & f \text{ surjective} \end{array}$$

Need to prove
f surjective $\Rightarrow$ f injective

This is the same as
f <u>not</u> injective $\Rightarrow$ f <u>not</u> surjective

f not injective $\Rightarrow |\mathrm{Im}(f)| < |S| \Rightarrow |\mathrm{Im}(f)| < |T| \Rightarrow \mathrm{Im}(f) \neq T$

2. let S and T be sets. If $U \subset S$ and $V \subset T$ then $U \times V \subset S \times T$. Are all subsets of $S \times T$ of this form? Carefully justify your answer.

Solution: No!

Observe that $(x, y) \in U \times V \iff x \in U, y \in V$.

Let $S = T = \mathbb{R}$, $W = \{(x,x) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$

If $W = U \times V \Rightarrow (x,x) \in U \times V \ \forall x \in \mathbb{R} \Rightarrow x \in U, x \in V \ \forall x \in \mathbb{R}$

$\Rightarrow U = V = \mathbb{R} \Rightarrow W = \mathbb{R} \times \mathbb{R}$. Contradiction

3. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $S_1 = \{1, 2, 3, 4\}$, $S_2 = \{3, 5, 6\}$, $S_3 = \{1, 3\}$. Define the relation (on S)

$$n \sim m \iff n, m \in S_i \text{ for some } i \in \{1, 2, 3\}.$$

Does this give an equivalence relation? Carefully justify your answer.

Solution:

No! Transitivity does not hold. For example $\underbrace{1 \sim 3}_{S_3}, \underbrace{3 \sim 5}_{S_2}$, but $1 \not\sim 5$

4. Is the function

$$f : \mathbb{Z} \to \mathbb{Z}$$
$$x \to x^3 + 3x$$

injective? Is it surjective? Carefully justify you answer.

Solution:

Consider $\phi : \mathbb{R} \to \mathbb{R}$
$x \longrightarrow x^3 + 3x$

$$\frac{d\phi}{dx} = 3x^2 + 3 = 3(x^2+1) > 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow \phi$ injective by the mean value theorem. $\left(\begin{matrix} \phi(x) = \phi(y) \\ x < y \end{matrix} \Rightarrow \begin{matrix} \exists c \in [x,y] \\ \text{with} \\ \phi'(c) = 0 \end{matrix} \right)$

$\Rightarrow f$ injective

cubic

$\phi(x) = x^3 + 3x \Rightarrow \phi$ surjective

However f is not surjective. For example

$f(1) = 4$

$f(2) = 14$

Because ϕ is monotonic increasing, and $(1,2)$ contains no integers $\nexists\, x \in \mathbb{Z}$ such that $f(x) = 5$.